

RESEARCH PAPER: 1993-3

A GENERAL DIFFERENTIAL DEMAND MODEL FOR MIXED QUANTITY- AND PRICE-DEPENDENT EQUATIONS

BY

Mark G. Brown, Sr. Research Economist -- FDOC

Jonq-Ying Lee, Sr. Research Economist – FDOC

Xiaoming Gao, Research Economist – Univ. of Arkansas

FLORIDA DEPARTMENT OF CITRUS

Economic and Market Research Department

P.O. Box 110249

Gainesville, Florida 32611-2049 USA

Phone: 352-392-1874

Fax: 352-392-8634

Email: mgbrown@ufl.edu

www.floridajuce.com

4/20/93

SK 93-12

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Mark G. Brown
Jonq-Ying Lee
Xiaoming Gao**

****Mark G. Brown and Jonq-Ying Lee are Research Economists, Florida Department of Citrus, and Courtesy Associate Professor and Courtesy Professor, respectively, Department of Food and Resource Economics, University of Florida, Gainesville, Florida 32611; Xiaoming Gao is a Research Economist, Department of Agricultural Economics and Rural Sociology, University of Arkansas, Fayetteville, Arkansas 72701.**

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Abstract

A mixed demand system is developed using the fundamental matrix equation of demand theory. The model parameterization combines the parameterizations of the quantity- and price-dependent Rotterdam models. The relationship between the mixed-demand-model parameters and the underlying direct and indirect utility functions allows analysis of various separability assumptions.

Key words: mixed demand, separability, Rotterdam models.

A General Differential Demand Model for Mixed Quantity- and Price-Dependent Equations

Demand systems are usually estimated assuming either prices and income are predetermined or quantities are predetermined. The former assumption leads to direct quantity-dependent demand specifications, while the latter leads to indirect or inverse price-dependent specifications. However, in empirical research, one may find it more appropriate to treat prices for some goods and quantities for the remaining goods as predetermined. For such cases, a mixed demand system might be considered.

The mixed demand problem has been studied by Samuelson, Heien, and Chavas, among others. Recent studies by Barten (1992), and Moschini and Vissa show how mixed demand might be applied empirically. The Barten study shows how to estimate mixed demand parameters using the maximum likelihood method—the maximum likelihood method is used to estimate the direct differential (Rotterdam) demand model, appropriately treating prices and quantities as either exogenous or endogenous; the direct demand parameter estimates for the Rotterdam model can then be transformed to obtain mixed demand parameters. The Moschini and Vissa application is based on a mixed differential demand specification and uses the restricted cost function to derive symmetry parameter restrictions. Although Moschini and Vissa's approach provides some useful restrictions, one might be interested in additional restrictions such as those stemming from separability and the structure of preferences. Barten's approach can be used to introduce restrictions related to the form of the direct utility function, as the Slutsky coefficients of the Rotterdam model can be directly related to the Hessian matrix of the utility maximization problem through Barten's (1964) fundamental equation. Weak and strong separability of the direct utility function might be assumed, providing useful restrictions on the Slutsky coefficients

(Theil, 1976). However, the underlying utility function of the mixed demand problem involves both the direct and indirect utility functions, and one may be interested in restrictions other than those related to the direct utility function which are usually imposed on the Rotterdam model.

The purpose of this paper is to examine the mixed demand problem under various separability assumptions and develop a general Rotterdam model which allows mixed separability restrictions associated with both the direct and indirect utility functions. The general model developed includes the direct and inverse Rotterdam demand specifications as special cases. The next section reviews the consumer choice problem and develops the fundamental equation for the mixed demand model. A decomposition of the price and quantity effects for the mixed demand model is obtained and used to parameterize a differential specification of demand. A levels version of the differential mixed demand model is also suggested. Basic separability restrictions for the mixed demand parameterization are then discussed, and, in the last section, elasticity formulas are provided.

Consumer-Choice Problem for Mixed Demand

Following Samuelson and Chavas, the consumer-choice problem for mixed demand can be written as

$$(1) \quad \begin{aligned} &\underset{q_a, \pi_a}{\text{maximize}} && u(q_a, q_b) - \psi(\pi_a, \pi_b) \\ &\text{subject to} && \pi_a' q_a + \pi_b' q_b = 1 \end{aligned}$$

where $u(\cdot)$ and $\psi(\cdot)$ are the direct and indirect utility functions, respectively; q_a and π_a are vectors of quantities and normalized prices for group a ; and q_b and π_b are vectors of quantities and normalized prices for group b . Normalized prices are actual prices p divided

by total expenditure or income x ; i.e., $\pi_a = \frac{P_a}{x}$ and $\pi_b = \frac{P_b}{x}$. Prices in group a are assumed to

be predetermined while quantities in group b are predetermined. The direct utility and indirect utility functions are assumed to be quasi-concave and quasi-convex, respectively. (As discussed by Chavas and Samuelson, the curvature assumptions may not always result in a unique solution for the maximization problem. In that case, we focus on the domain of q_a and π_b , where the solution to (1) is unique.) At the optimum, $u(q_a, q_b) - \psi(\pi_a, \pi_b) = 0$.

The first order conditions for (1) are

$$(2) \quad \frac{\partial u}{\partial q_a} = \lambda \pi_a$$

$$-\frac{\partial \psi}{\partial \pi_b} = \lambda q_b$$

$$\pi'_a q_a + \pi'_b q_b = 1,$$

where λ is the Lagrange multiplier and $\frac{\partial u}{\partial q_a}$ and $-\frac{\partial \psi}{\partial \pi_b}$ are vectors of partial derivatives of the

direct and indirect utility functions with respect to quantities and normalized prices, respectively.

The solution to (2) is

$$(3) \quad q_a = q_a(\pi_a, q_b)$$

$$\pi_b = \pi_b(\pi_a, q_b)$$

$$\lambda = \lambda(\pi_a, q_b).$$

The equations $q_a = q_a(\pi_a, q_b)$ and $\pi_b = \pi_b(\pi_a, q_b)$ comprise the mixed demand system examined in this study.

Fundamental Equation

The next step of our analysis is to develop the fundamental equation for the mixed demand problem (the counterpart to Barten's fundamental equation for direct demand) and determine the effects of prices and quantities in terms of the underlying utility functions.

Total differentiation of first-order condition (2) results in

$$(4) \quad \begin{bmatrix} U_{aa} & 0 & \pi_a \\ 0 & \psi_{bb} & q_b \\ \pi'_a & q'_b & 0 \end{bmatrix} \begin{bmatrix} dq_a \\ d\pi_b \\ -d\lambda \end{bmatrix} = \begin{bmatrix} \lambda I & -U_{ab} \\ -\psi_{ab} & \lambda I \\ -q'_a & -\pi'_b \end{bmatrix} \begin{bmatrix} d\pi_a \\ dq_b \end{bmatrix},$$

where second partial derivative matrices are defined by $U_{aa} = \frac{\partial^2 u}{\partial q_a \partial q'_a}$, $U_{ab} = \frac{\partial^2 u}{\partial q_a \partial q'_b}$,

$$\psi_{bb} = -\frac{\partial \psi}{\partial \pi_b \partial \pi'_b}, \text{ and } \psi_{ab} = -\frac{\partial \psi}{\partial \pi_b \partial \pi'_a}.$$

Likewise, total differentiation of solution (3) yields

$$(5) \quad \begin{bmatrix} dq_a \\ d\pi_b \\ -d\lambda \end{bmatrix} = \begin{bmatrix} \frac{\partial q_a}{\partial \pi'_a} & \frac{\partial q_a}{\partial q'_b} \\ \frac{\partial \pi_b}{\partial \pi'_a} & \frac{\partial \pi_b}{\partial q'_b} \\ -\frac{\partial \lambda}{\partial \pi'_a} & -\frac{\partial \lambda}{\partial q'_b} \end{bmatrix} \begin{bmatrix} d\pi_a \\ dq_b \end{bmatrix}.$$

Combining (4) and (5), we see

$$(6) \begin{bmatrix} U_{aa} & 0 & \pi_a \\ 0 & \psi_{bb} & q_b \\ \pi'_a & q'_b & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial q_a}{\partial \pi'_a} & \frac{\partial q_a}{\partial q'_b} \\ \frac{\partial \pi_b}{\partial \pi'_a} & \frac{\partial \pi_b}{\partial q'_b} \\ -\frac{\partial \lambda}{\partial \pi'_a} & -\frac{\partial \lambda}{\partial q'_b} \end{bmatrix} = \begin{bmatrix} \lambda I & -U_{ab} \\ -\psi_{ab} & \lambda I \\ -q'_a & -\pi'_b \end{bmatrix},$$

$$\text{or} \begin{bmatrix} \frac{\partial q_a}{\partial \pi'_a} & \frac{\partial q_a}{\partial q'_b} \\ \frac{\partial \pi_b}{\partial \pi'_a} & \frac{\partial \pi_b}{\partial q'_b} \\ -\frac{\partial \lambda}{\partial \pi'_a} & -\frac{\partial \lambda}{\partial q'_b} \end{bmatrix} = \begin{bmatrix} U_{aa} & 0 & \pi_a \\ 0 & \psi_{bb} & q_b \\ \pi'_a & q'_b & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda I & -U_{ab} \\ -\psi_{ab} & \lambda I \\ -q'_a & -\pi'_b \end{bmatrix}.$$

Applying the rule for inverting a partitioned matrix, the price and quantity effects for mixed demand equations and λ are

$$(7) \begin{bmatrix} \frac{\partial q_a}{\partial \pi'_a} & \frac{\partial q_a}{\partial q'_b} \\ \frac{\partial \pi_b}{\partial \pi'_a} & \frac{\partial \pi_b}{\partial q'_b} \\ -\frac{\partial \lambda}{\partial \pi'_a} & -\frac{\partial \lambda}{\partial q'_b} \end{bmatrix} = \phi^{-1} \begin{bmatrix} \phi \begin{bmatrix} U_{aa}^{-1} & 0 \\ 0 & \Psi_{bb}^{-1} \end{bmatrix} - \begin{bmatrix} U_{aa}^{-1} & 0 \\ 0 & \Psi_{bb}^{-1} \end{bmatrix} \begin{bmatrix} \pi_a \\ q_b \end{bmatrix} \begin{bmatrix} \pi'_a & q'_b \end{bmatrix} \begin{bmatrix} U_{aa}^{-1} & 0 \\ 0 & \Psi_{bb}^{-1} \end{bmatrix} & \begin{bmatrix} U_{aa}^{-1} & 0 \\ 0 & \Psi_{bb}^{-1} \end{bmatrix} \begin{bmatrix} \pi_a \\ q_b \end{bmatrix} \\ \begin{bmatrix} \pi'_a & q'_b \end{bmatrix} \begin{bmatrix} U_{aa}^{-1} & 0 \\ 0 & \Psi_{bb}^{-1} \end{bmatrix} & -1 \end{bmatrix} \begin{bmatrix} \lambda I & -U_{ab} \\ -\psi_{ab} & \lambda I \\ -q'_a & -\pi'_b \end{bmatrix},$$

$$= \begin{bmatrix} \begin{bmatrix} \lambda U_{aa}^{-1} \\ -\Psi_{bb}^{-1} \Psi_{ab} \end{bmatrix} - \lambda \phi^{-1} \begin{bmatrix} U_{aa}^{-1} \pi_a \pi_a' U_{aa}^{-1} \\ \Psi_{bb}^{-1} q_b \pi_a' U_{aa}^{-1} \end{bmatrix} + \phi^{-1} \begin{bmatrix} U_{aa}^{-1} \pi_a q_b' \Psi_{bb}^{-1} \Psi_{ab} \\ \Psi_{bb}^{-1} q_b q_b' \Psi_{bb}^{-1} \Psi_{ab} \end{bmatrix} - \phi^{-1} \begin{bmatrix} U_{aa}^{-1} \pi_a q_a' \\ \Psi_{bb}^{-1} q_b q_a' \end{bmatrix} \\ \phi^{-1} (\lambda \pi_a' U_{aa}^{-1} - q_b' \Psi_{bb}^{-1} \Psi_{ab} + q_a') \\ \begin{bmatrix} -U_{aa}^{-1} U_{ab} \\ \lambda \Psi_{bb}^{-1} \end{bmatrix} + \phi^{-1} \begin{bmatrix} U_{aa}^{-1} \pi_a \pi_a' U_{aa}^{-1} U_{ab} \\ \Psi_{bb}^{-1} q_b \pi_a' U_{aa}^{-1} U_{ab} \end{bmatrix} - \lambda \phi^{-1} \begin{bmatrix} U_{aa}^{-1} \pi_a q_b' \Psi_{bb}^{-1} \\ \Psi_{bb}^{-1} q_b q_b' \Psi_{bb}^{-1} \end{bmatrix} - \phi^{-1} \begin{bmatrix} U_{aa}^{-1} \pi_a \pi_b' \\ \Psi_{bb}^{-1} q_b \pi_b' \end{bmatrix} \\ \phi^{-1} (-\pi_a' U_{aa}^{-1} U_{ab} + \lambda q_b' \Psi_{bb}^{-1} + \pi_b') \end{bmatrix},$$

where $\phi^{-1} = (\pi_a' U_{aa}^{-1} \pi_a + q_b' \Psi_{bb}^{-1} q_b)^{-1}$. Result (7) is the fundamental equation for the mixed demand problem.

Income and Scale Effects and General Restrictions

The income effects for the model are

$$(8) \quad \begin{bmatrix} x \frac{\partial q_a}{\partial x} \\ x \frac{\partial \pi_b}{\partial x} \\ x \frac{\partial \lambda}{\partial x} \end{bmatrix} = - \begin{bmatrix} \frac{\partial q_a}{\partial \pi_a'} \\ \frac{\partial \pi_b}{\partial \pi_a'} \\ \frac{\partial \lambda}{\partial \pi_a'} \end{bmatrix} \pi_a,$$

while the scale effects are

$$(9) \quad \begin{bmatrix} k \frac{\partial q_a}{\partial k} \\ k \frac{\partial \pi_b}{\partial k} \\ k \frac{\partial \lambda}{\partial k} \end{bmatrix} = \begin{bmatrix} \frac{\partial q_a}{\partial q'_b} \\ \frac{\partial \pi_b}{\partial q'_b} \\ \frac{\partial \lambda}{\partial q'_b} \end{bmatrix} q_b,$$

where k is a scalar measuring the size of the bundle of goods in group b ; i.e., $q_b = k q_b^0$ where q_b^0 is a reference bundle. Letting the bundle q_b and the reference bundle q_b^0 coincide, k equals one.

From the budget constraint $\pi'_a q_a(\pi_a, q_b) + \pi_b(\pi_a, q_b)' q_b = 1$, the adding-up restrictions for (7) are

$$(10a) \quad \pi'_a \frac{\partial q_a}{\partial \pi'_a} + q'_a + q'_b \frac{\partial \pi_b}{\partial \pi'_a} = 0$$

and

$$(10b) \quad \pi'_a \frac{\partial q_a}{\partial q'_b} + q'_b \frac{\partial \pi_b}{\partial q'_b} + \pi'_b = 0.$$

Differentiating the budget constraint with respect to income x and scale k (for bundle q_b) also yields essentially the same adding-up restrictions, i.e.,

$$(10c) \quad -\pi'_a \frac{\partial q_a}{\partial \pi'_a} \frac{\pi_a}{x} - q'_b \frac{\partial \pi_b}{\partial \pi'_a} \frac{\pi_a}{x} - \frac{\pi'_a q_a}{x} = 0$$

$$(10d) \quad \pi'_a \frac{\partial q_a}{\partial q'_b} q_b + q'_b \frac{\partial \pi_b}{\partial q'_b} q_b + \pi'_b q_b = 0.$$

Homogeneity with respect to prices and income is satisfied given the normalization of prices. Symmetry restrictions will be discussed subsequently in context of our differential demand parameterization for the mixed demand problem.

Differential Mixed Demand Specification

We choose to parameterize the mixed demand model in terms of conditional income and scale effects. Conditional direct demand equations with some quantities predetermined and conditional inverse demand equations with some prices predetermined are considered, following Chavas. The conditional direct demand equations for q_a , given π_a , π_b and q_b are predetermined, have conditional income effects

$$(11) \quad \frac{\partial q_a}{\partial x} = \frac{1}{x} \left(\pi_a' U_{aa}^{-1} \pi_a \right)^{-1} U_{aa}^{-1} \pi_a,$$

while the conditional inverse demand equations for π_b , given q_a , q_b and π_a are predetermined, have conditional scale effects

$$(12) \quad \frac{\partial \pi_b}{\partial k} = - \left(q_b' \psi_{bb}^{-1} q_b \right)^{-1} \psi_{bb}^{-1} q_b.$$

The derivations for conditional effects (11) and (12) are provided in Appendix A. Notice that the conditional income and scale effects, outside of the terms $\left(\pi_a' U_{aa}^{-1} \pi_a \right)^{-1}$ and $\left(q_b' \psi_{bb}^{-1} q_b \right)^{-1}$, appear in equation (7). The parameterization developed focuses on this relationship.

Directing our attention on individual matrix elements, the total differential of the mixed demand equations $q_a(\pi_a, q_b)$ and $\pi_b(\pi_a, q_b)$ can be written as

$$(13) \quad dq_{ai} = \sum_j \frac{\partial q_{ai}}{\partial \pi_{aj}} d\pi_{aj} + \sum_s \frac{\partial q_{ai}}{\partial q_{bs}} dq_{bs}$$

and

$$(14) \quad d\pi_{br} = \sum_j \frac{\partial \pi_{br}}{\partial \pi_{aj}} d\pi_{aj} + \sum_s \frac{\partial \pi_{br}}{\partial q_{bs}} dq_{bs},$$

where subscripts i, j, r and s have been attached to indicate elements of the vectors or matrices involved. Multiplying (13) by π_{ai} and (14) by q_{br} results in

$$(15) \quad \pi_{ai} q_{ai} \frac{dq_{ai}}{q_{ai}} = \sum_j \pi_{ai} \pi_{aj} \frac{\partial q_{ai}}{\partial \pi_{aj}} \frac{d\pi_{aj}}{\pi_{aj}} + \sum_s \pi_{ai} q_{bs} \frac{\partial q_{ai}}{\partial q_{bs}} \frac{dq_{bs}}{q_{bs}}$$

or

$$w_{ai} d \log q_{ai} = \sum_j \pi_{ai} \pi_{aj} \frac{\partial q_{ai}}{\partial \pi_{aj}} d \log \pi_{aj} + \sum_s \pi_{ai} q_{bs} \frac{\partial q_{ai}}{\partial q_{bs}} d \log q_{bs}$$

and

$$(16) \quad q_{br} \pi_{br} \frac{d\pi_{br}}{\pi_{br}} = \sum_j q_{br} \pi_{aj} \frac{\partial \pi_{br}}{\partial \pi_{aj}} \frac{d\pi_{aj}}{\pi_{aj}} + \sum_s q_{br} q_{bs} \frac{\partial \pi_{br}}{\partial q_{bs}} \frac{dq_{bs}}{q_{bs}}$$

or

$$w_{br} d \log \pi_{br} = \sum_j q_{br} \pi_{aj} \frac{\partial \pi_{br}}{\partial \pi_{aj}} d \log \pi_{aj} + \sum_s q_{br} q_{bs} \frac{\partial \pi_{br}}{\partial q_{bs}} d \log q_{bs}$$

where $w_{ai} = \frac{P_{ai} q_{ai}}{x}$ and $w_{bs} = \frac{P_{bs} q_{bs}}{x}$, the budget shares for good i in group a and good s in

group b , respectively.

Next, we replace the partial derivatives on the right sides of (15) and (16) with the corresponding matrix elements in (7), preserving the conditional income and scale components embedded in the decomposition, i.e.,

$$\begin{aligned}
 (17) \quad w_{ai} d \log q_{ai} = & \sum_j \left[\lambda \pi_{ai} \pi_{aj} U_{aa}^{ij} - \lambda \phi^{-1} \pi_{ai} \left[\sum_h U_{aa}^{ih} \pi_{ah} \right] \pi_{aj} \left[\sum_h \pi_{ah} U_{aa}^{hj} \right] \right. \\
 & + \phi^{-1} \pi_{ai} \left(U_{aa}^{ih} \pi_{ah} \right) \sum_s \left[\sum_g q_{bg} \psi_{bb}^{gs} \right] \lambda q_{bs} (\lambda q_{bs})^{-1} \pi_{aj} \psi_{ab}^{sj} - \phi^{-1} \pi_{ai} \left[\sum_h U_{aa}^{ih} \pi_{ah} \right] \pi_{aj} q_{aj} \left. \right] d \log \pi_{aj} \\
 & + \sum_s \left[\left[- \sum_j U_{aa}^{ij} \lambda \pi_{aj} \pi_{ai} (\lambda \pi_{aj})^{-1} q_{bs} U_{ab}^{js} \right] + \phi^{-1} \pi_{ai} \left[\sum_h U_{aa}^{ih} \pi_{ah} \right] \sum_j \sum_h \pi_{ah} U_{aa}^{hj} \lambda \pi_{aj} (\lambda \pi_{aj})^{-1} q_{bs} U_{ab}^{js} \right. \\
 & - \lambda \phi^{-1} \pi_{ai} \left[\sum_h U_{aa}^{ih} \pi_{ah} \right] q_{bs} \left[\sum_g q_{bg} \psi_{bb}^{gs} \right] - \phi^{-1} \pi_{ai} \left[\sum_h U_{aa}^{ih} \pi_{ah} \right] \pi_{bs} q_{bs} \left. \right] d \log q_{bs} \\
 w_{br} d \log \pi_{br} = & \sum_j \left[\left[- \sum_s \psi_{bb}^{rs} \lambda q_{bs} q_{br} (\lambda q_{bs})^{-1} \pi_{aj} \psi_{ab}^{sj} \right] - \lambda \phi^{-1} q_{br} \left[\sum_s \psi_{bb}^{rs} q_{bs} \right] \pi_{aj} \left[\sum_h \pi_{ah} U_{aa}^{hj} \right] \right. \\
 & + \phi^{-1} q_{br} \left[\sum_s \psi_{bb}^{rs} q_{bs} \right] \sum_s \left[\sum_g q_{bg} \psi_{bb}^{gs} \right] \lambda q_{bs} (\lambda q_{bs})^{-1} \pi_{aj} \psi_{ab}^{sj} - \phi^{-1} q_{br} \left[\sum_s \psi_{bb}^{rs} q_{bs} \right] \pi_{aj} q_{aj} \left. \right] d \log \pi_{aj} \\
 & + \sum_s \left[\left[\lambda q_{br} q_{hs} \psi_{bb}^{rs} + \phi^{-1} q_{br} \left[\sum_s \psi_{bb}^{rs} q_{bs} \right] \sum_j \sum_h \pi_{ah} U_{aa}^{hj} \lambda \pi_{aj} (\lambda \pi_{aj})^{-1} q_{hs} U_{ab}^{js} \right. \right. \\
 & \left. \left. - \lambda \phi^{-1} q_{br} \left[\sum_s \psi_{bb}^{rs} q_{bs} \right] q_{bs} \left[\sum_g q_{bg} \psi_{bb}^{gs} \right] - \phi^{-1} q_{br} \left[\sum_s \psi_{bb}^{rs} q_{bs} \right] \pi_{bs} q_{bs} \right] d \log q_{bs}, \right.
 \end{aligned}$$

where U_{aa}^{ij} is the ij^{th} element of U_{aa}^{-1} ; U_{ab}^{hs} is the hs^{th} element of U_{ab} ; ψ_{bb}^{gs} is the gs^{th} element of

ψ_{bb}^{-1} ; and ψ_{ab}^{sj} is the sj^{th} element of ψ_{ab} . The following notation is used to write (17) more

compactly:

$$(18) \quad \theta_i = \phi^{-1} \pi_{ai} \sum_h U_{aa}^{ih} \pi_{ah} = \phi^{-1} \phi_1 \pi_{ai} x \frac{\partial q_{ai}}{\partial x} = \phi^{-1} \phi_1 p_{ai} \frac{\partial q_{ai}}{\partial x}$$

$$\phi_1 = \pi_a' U_{aa}^{-1} \pi_a$$

$$\theta_{ij} = \phi^{-1} \pi_{ai} \pi_{aj} U_{aa}^{ij}$$

$$\gamma_r = \phi^{-1} q_{br} \sum_s \psi_{bb}^{rs} q_{bs} = -\phi^{-1} \phi_2 q_{br} \frac{\partial \pi_{br}}{\partial k}$$

$$\phi_2 = q_b' \psi_{bb}^{-1} q_b$$

$$\gamma_{rs} = \phi^{-1} q_{br} q_{bs} \psi_{bb}^{rs}$$

$$U_{ab}^{*js} = \frac{\partial \left[\frac{\partial U}{\partial q_{aj}} \right]}{\partial q_{bs}} \frac{q_{bs}}{\lambda \pi_{aj}}$$

$$\Psi_{ab}^{*sj} = \frac{\partial \left[\frac{\partial \Psi}{\partial \pi_{bs}} \right]}{\partial \pi_{aj}} \frac{\pi_{aj}}{\lambda \pi_{bs}},$$

where $\frac{\partial q_{ai}}{\partial x}$ and $\frac{\partial \pi_{br}}{\partial k}$ are the conditional income and scale effects as defined in (11) and (12); U_{ab}^{*js}

is the elasticity of marginal utility of good j in group a with respect to a change in good s in group b ; and Ψ_{ab}^{*sj} is the elasticity of marginal utility of normalized price s in group b with respect

to a change in normalized price j in group a . Notice that:

$$\phi_1 + \phi_2 = \phi \quad \text{or} \quad \frac{\phi_1}{\phi} + \frac{\phi_2}{\phi} = 1$$

$$\sum_i \theta_i = \frac{\phi_1}{\phi} \quad \sum_r \gamma_r = \frac{\phi_2}{\phi}$$

$$\sum_i \theta_{ij} = \theta_j \quad \sum_r \gamma_{rs} = \gamma_s$$

$$\sum_j \theta_{ij} = \theta_i \quad \sum_s \gamma_{rs} = \gamma_r$$

$$\theta_{ji} = \theta_{ij} \quad \gamma_{rs} = \gamma_{sr}.$$

Using the latter notation, (17) can be written as

$$\begin{aligned}
 (19) \quad w_{ai} d \log q_{ai} &= -\theta_i \left(\sum_j w_{aj} d \log \pi_{aj} + \sum_s w_{bs} d \log q_{bs} \right) \\
 &\quad + \lambda \phi \sum_j \left[\theta_{ij} - \theta_i \theta_j + \theta_i \sum_s \gamma_s \psi_{ab}^{*sj} \right] d \log \pi_{aj} \\
 &\quad + \lambda \phi \sum_s \left[-\sum_j \theta_{ij} U_{ab}^{*js} + \theta_i \sum_j \theta_j U_{ab}^{*js} - \theta_i \gamma_s \right] d \log q_{bs} \\
 w_{br} d \log \pi_{br} &= -\gamma_r \left(\sum_j w_{aj} d \log \pi_{aj} + \sum_s w_{bs} d \log q_{bs} \right) \\
 &\quad + \lambda \phi \sum_j \left[-\sum_s \gamma_{rs} \psi_{ab}^{*sj} - \gamma_r \theta_j + \gamma_r \sum_s \gamma_s \psi_{ab}^{*sj} \right] d \log \pi_{aj} \\
 &\quad + \lambda \phi \sum_s \left[\gamma_{rs} + \gamma_r \sum_j \theta_j U_{ab}^{*js} - \gamma_r \gamma_s \right] d \log q_{bs}.
 \end{aligned}$$

Summing (19) over i in group a , and r in group b , and using the results from (18), one has

$$(20) \quad \sum_i w_{ai} d \log q_{ai} + \sum_r w_{br} d \log \pi_{br} = -\sum_i w_{ai} d \log \pi_{ai} - \sum_r w_{br} d \log q_{br},$$

which shows our specification adds up as (20) is just the total differential of the budget constraint

$$\pi'_a q_a + \pi'_b q_b = 1.$$

In specification (19), symmetry involves $\theta_{ij} = \theta_{ji}$ and $\gamma_{rs} = \gamma_{sr}$ and follows directly from the symmetry of second partial utility matrices involved. As mentioned earlier, homogeneity is satisfied as prices are normalized. As shown in (18), θ_i is the conditional marginal propensity to consume weighted by $\frac{\phi_1}{\phi}$ and γ_r is the conditional scale coefficient for inverse demand

weighted by $1 - \frac{\phi_1}{\phi} = \frac{\phi_2}{\phi}$. Except for factors of proportionality, the terms θ_i and γ_r have the

same form as the income and scale coefficients of the direct and inverse Rotterdam models, respectively, while θ_{ij} and γ_{rs} are corresponding substitution terms (these relationships are further discussed in the section on "Direct versus Inverse Rotterdam Models").

Mixed Income

In the Rotterdam model, the income variable is the Divisia volume index. Ignoring the two groups for the mixed demand problem, the budget constraint $x = \sum_i p_i q_i$ can be totally

differentiated to obtain

$$(21) \quad d \log x = d \log P + d \log Q$$

where $d \log P = \sum_i w_i d \log p_i$ is the Divisia price index and $d \log Q = \sum_i w_i d \log q_i$ is the

Divisia volume index.

From (21), the Divisia volume index can be written as

$$(22) \quad d \log Q = d \log x - d \log P$$

$$= d \log \frac{x}{P},$$

and can be viewed as a measure of the change in real income or deflated nominal income

$$\left[\frac{x}{P} \right].$$

For the direct Rotterdam model, the dependent variable, which measures the change in quantity demanded, is $w_i d \log q_i$ and is the contribution made by good i to the Divisia volume index or the change in real income.

For the inverse Rotterdam model, the dependent variable, which measures the change in real prices, is $w_i d \log \pi_i = -w_i d \log \frac{x}{p_i} = -w_i d \log q_i^m$, where q_i^m is the maximum amount of

good i that could be purchased with income x . Summing the latter variable over i , we see

$$(23) \quad \sum w_i d \log \pi_i = -(d \log x - d \log P) \\ = -d \log Q.$$

Hence, the dependent variable for the inverse Rotterdam model can also be viewed as a contribution to the change in real income.

Although the sum of $w_i d \log q_i$ over i equals the negative of the sum of $w_i d \log \pi_i$ over i equals the Divisia volume index, the contributions made by good i to the change in real income in the direct and inverse Rotterdam models are clearly not the same ($w_i d \log q_i$ versus $w_i d \log \pi_i = d \log q_i^m$). That is, there are two measures of contributions to the change in real income—a budget-share-weighted real quantity change and a budget-share-weighted real price change.

For the mixed demand problem, both measures of contributions to the change in real income are utilized. When we have groups a and b , total differentiation of the budget constraint

$$1 = \pi'_a q_a + \pi'_b q_b \text{ yields}$$

$$(24) \quad 0 = d \log Q_a + d \log \Pi_a + d \log Q_b + d \log \Pi_b,$$

where $d \log Q_a = \sum_i w_{ai} d \log q_{ai}$, $d \log \Pi_a = \sum_i w_{ai} d \log \pi_{ai}$, $d \log Q_b = \sum_r w_{br} d \log q_{br}$ and

$$d \log \Pi_b = \sum_r w_{br} d \log \pi_{br}.$$

For the mixed demand model, the dependent variables are $w_{ai} d \log q_{ai}$ and $w_{br} d \log \pi_{br}$, and the sum of these dependent variables (over i and r) is $d \log Q_a + d \log \Pi_b$ $= -(d \log \Pi_a + d \log Q_b)$. The term $d \log Q_a + d \log \Pi_b$ can be viewed as a mixed measure of the change in income. The first part of this mixed measure of income, $d \log Q_a$, follows the definition of the change in real income in the direct Rotterdam model while the second part, $d \log \Pi_b$, follows the definition in the inverse Rotterdam model. However, the mixed measure of the change in income is not equal to the Divisia volume index or change in real income ($d \log Q_a + d \log Q_b \neq d \log Q_a + d \log \Pi_b$). This point will be discussed further when compensated effects for mixed demand equations are considered.

In the present analysis, the change in income in the mixed demand model will be measured by $d \log Q_a + d \log \Pi_b$ which can be used in place of $-(d \log \Pi_a + d \log Q_b)$ on the right side of equation (19). The latter substitution is preferred in empirical analysis as it ensures adding-up will be satisfied when discrete data are used and first differences are taken (for essentially the same substitution in the direct Rotterdam model, see Theil, 1971).

Direct and Inverse Rotterdam Models

Demand system (19) can be viewed as a general Rotterdam model with the direct and inverse Rotterdam models as special cases. When all (normalized) prices are predetermined, the second equation in (19) vanishes and the first equation becomes the relative price equation for the direct Rotterdam model (Theil, 1976), i.e.,

$$\begin{aligned} w_i d \log q_i &= -\theta_i \sum_j w_j d \log \pi_j + \lambda \phi \sum_j (\theta_{ij} - \theta_i \theta_j) d \log \pi_j \\ &= \theta_i \sum_j w_j d \log q_j + \phi^* \sum_j (\theta_{ij} - \theta_i \theta_j) d \log p_j, \end{aligned}$$

where the subscript for group a has been dropped and

$$\begin{aligned} \sum_j w_j d \log \pi_j &= \sum_j w_j (d \log p_j - d \log x) \\ &= -\sum_j w_j d \log q_j; \end{aligned}$$

$$\begin{aligned} \sum_j (\theta_{ij} - \theta_i \theta_j) d \log \pi_j &= \sum_j (\theta_{ij} - \theta_i \theta_j) (d \log p_j - d \log x) \\ &= \sum_j (\theta_{ij} - \theta_i \theta_j) d \log p_j \end{aligned}$$

as $\sum_j (\theta_{ij} - \theta_i \theta_j) = 0$; and $\phi^* = \lambda \phi = \left[\frac{\partial \log \lambda^*}{\partial \log x} \right]^{-1}$ where $\lambda = \lambda^* x$ and λ^* is the Lagrange

multiplier for the maximization problem when prices are not normalized. The terms $\phi^* \theta_{ij}$ and $\phi^* \theta_i \theta_j$ are known as the specific and general substitution parameters and the overall term $\phi^* (\theta_{ij} - \theta_i \theta_j)$ is the Slutsky coefficient (Theil, 1976).

On the other hand, when all quantities are predetermined, the first equation in (19) vanishes and the second equation becomes the Rotterdam inverse demand model (Barten and Bettendorf), i.e.,

$$\begin{aligned} w_r d \log \pi_r &= -\gamma_r \sum_s d \log q_s + \lambda \phi \sum_s (\gamma_{rs} - \gamma_r \gamma_s) d \log q_s \\ &= \gamma_r^* \sum_s d \log q_s + \sum_s \gamma_{rs}^* d \log q_s, \end{aligned}$$

where subscript b has been dropped and $\gamma_r^* = -\gamma_r = q_r \frac{\partial \pi_r}{\partial k}$; $\gamma_{rs}^* = \lambda \phi (\gamma_{rs} - \gamma_r \gamma_s)$. Notice

that Barten and Bettendorf do not decompose γ_{rs}^* , referred to as the Antonelli coefficient.

However, one may be interested in the decomposition shown above since γ_{rs} is directly related to the indirect utility function, allowing separability restrictions to be explicitly considered.

Levels Version of Mixed Demand

For some empirical studies, one may want a mixed demand system in terms of levels as opposed to differences. Differential demand models are primarily time series models. For cross-section data, demand specifications in terms of levels are desirable. Demand specifications in terms of levels may also be more amenable to capture dynamic relationships. Barten (1989) has recently developed a levels version for the Rotterdam model. Alternatively, one might consider Bewley and Elliot's Generalized Addilog Demand System (GADS) as a levels version of the Rotterdam model. In this paper, we develop a levels version for mixed demand model (19), following Barten.

Barten's levels version for the Rotterdam model is based on the income identity

$$\begin{aligned}
 (25) \quad \log x &= \sum_i w_i \log x \\
 &= \sum_i w_i \log \left[\frac{p_i q_i}{w_i} \right] \\
 &= \sum_i w_i \log q_i + \sum_i w_i \log p_i - \sum_i w_i \log w_i.
 \end{aligned}$$

As discussed by Barten, usually the last term on the right side of (25), $\sum_i w_i \log w_i$, varies little and can be reasonably approximated by a constant. In such cases, real income defined by $\log x - \sum_i w_i \log p_i$ can be measured by $\sum_i w_i \log q_i - c$, where c is the constant used to approximate $\sum_i w_i \log w_i$. That is, except for constant c , $\sum_i w_i \log q_i$ can be viewed as a measure of real income. Hence, Barten's levels version of the Rotterdam model is

$$(26) \quad w_i \log q_i = a_i + \theta_i \left(\sum_j w_j \log q_j \right) - \sum_j c_{ij} \log p_j,$$

where $c_{ij} = \phi^*(\theta_{ij} - \theta_i \theta_j)$. Using $\sum_i w_i \log q_i - c$ as real income instead of $\log x - \sum_i w_i \log p_i$ is desirable since it makes the model consistent with the adding-up property (the

sum of the left-side variables of equation (26) equals the real income variable $\sum_i w_i d \log q_i$).

For mixed demand, consider the system of demand equations in terms of levels

$$(27a) \quad \log q_{ai} = \beta_{ai} + \sum_j \varepsilon_{ai,aj} \log \pi_{aj} + \sum_s \varepsilon_{ai,bs} \log q_{bs}$$

$$(27b) \quad \log \pi_{br} = \beta_{br} + \sum_j \varepsilon_{br,aj} \log \pi_{aj} + \sum_s \varepsilon_{br,bs} \log q_{bs},$$

where β_{ai} and β_{br} are intercepts, and $\varepsilon_{ai,aj} = \frac{\partial \log q_{ai}}{\partial \log \pi_{aj}}$, $\varepsilon_{ai,bs} = \frac{\partial \log q_{ai}}{\partial \log q_{bs}}$, $\varepsilon_{br,aj} = \frac{\partial \log \pi_{br}}{\partial \log \pi_{aj}}$, and

$$\varepsilon_{br,bs} = \frac{\partial \log \pi_{br}}{\partial \log q_{bs}}.$$

A mixed demand model in levels, consistent with differential model (19), can be developed by first multiplying equations (27a) and (27b) through by their respective budget shares and rearranging terms to obtain an income variable, i.e.,

$$\begin{aligned} (28a) \quad w_{ai} \log q_{ai} &= w_{ai} \beta_{ai} - \theta_i \left(\sum_j w_{aj} \log \pi_{aj} + \sum_s w_{bs} \log q_{bs} \right) \\ &+ \sum_j (w_{ai} \varepsilon_{ai,aj} + \theta_i w_j) \log \pi_{aj} \\ &+ \sum_s (w_{ai} \varepsilon_{ai,bs} + \theta_i w_{bs}) \log q_{bs} \end{aligned}$$

$$\begin{aligned} (28b) \quad w_{br} \log \pi_{br} &= w_{br} \beta_{br} - \gamma_r \left(\sum_j w_{aj} \log \pi_{aj} + \sum_s w_{bs} \log q_{bs} \right) \\ &+ \sum_j (w_{br} \varepsilon_{br,aj} + \gamma_r w_{aj}) \log \pi_{aj} \\ &+ \sum_s (w_{br} \varepsilon_{br,bs} + \gamma_r w_{bs}) \log q_{bs}. \end{aligned}$$

By definition, we can write

$$0 = \sum_j w_{aj} \log \left[\frac{\pi_{aj} q_{aj}}{w_{aj}} \right] + \sum_s w_{bs} \log \left[\frac{\pi_{bs} q_{bs}}{w_{bs}} \right]$$

or

$$\log \Pi_a + \log Q_b = -\log Q_a - \log \Pi_b + \log W_a + \log W_b,$$

where $\log \Pi_a = \sum_j w_{aj} \log \pi_{aj}$, $\log Q_b = \sum_s w_{bs} \log q_{bs}$, $\log Q_a = \sum_j w_{aj} \log q_{aj}$, $\log \Pi_b =$

$$\sum_s w_{bs} \log \pi_{bs}, \log W_a = \sum_j w_{aj} \log w_{aj}, \text{ and } \log W_b = \sum_s w_{bs} \log w_{bs}.$$

As in the case of the levels version for the Rotterdam model, $\log W_a + \log W_b$ may vary little and be approximately equal to a constant c . In this case, we can substitute $c - \log Q_a - \log \Pi_b$ for $\log \Pi_a + \log Q_b$ in equations (28a) and (28b) to obtain a levels version of mixed demand consistent with adding-up, i.e.,

$$\begin{aligned} (29a) \quad w_{ai} \log q_{ai} = & \beta'_{ai} + \theta_i (\log Q_a + \log \Pi_b) \\ & + \lambda \phi \sum_j \left[\theta_{ij} - \theta_i \theta_j + \theta_i \sum_s \gamma_s \psi_{ab}^{*sj} \right] \log \pi_{aj} \\ & + \lambda \phi \sum_s \left[-\sum_j \theta_{ij} U_{ab}^{*js} + \theta_i \sum_j \theta_j U_{ab}^{*js} - \theta_i \gamma_s \right] \log q_{bs} \end{aligned}$$

$$\begin{aligned} (29b) \quad w_{br} \log \pi_{br} = & \beta'_{br} + \gamma_r (\log Q_a + \log \Pi_b) \\ & + \lambda \phi \sum_j \left[-\sum_s \gamma_{rs} \psi_{ab}^{*sj} - \gamma_r \theta_j + \gamma_r \sum_s \gamma_s \psi_{ab}^{*sj} \right] \log \pi_{aj} \\ & + \lambda \phi \sum_s \left[\gamma_{rs} + \gamma_r \sum_j \theta_j U_{ab}^{*js} - \gamma_r \gamma_s \right] \log q_{bs}. \end{aligned}$$

In equations (29a) and (29b), we use the same notation used in developing equation (19); i.e.,

$$w_{ai} \varepsilon_{ai,aj} + \theta_i w_j = \lambda \phi \left[\theta_{ij} - \theta_i \theta_j + \theta_i \sum_s \gamma_s \psi_{ab}^{*sj} \right], \text{ etc. (See equations (16) through (19).) The}$$

intercepts for (29a) and (29b) are $\beta'_{ai} = w_{ai} \beta_{ai} - \theta_i c$ and $\beta'_{br} = w_{br} \beta_{br} - \gamma_r c$.

Specific Restrictions for Mixed Demand

Focusing on models (19) and (29), consider introducing additional restrictions through the decomposition of the price and quantity effects.¹ In particular, one might be interested in separability restrictions related to the direct and indirect utility functions. For example, suppose groups a and b are strongly or additive separable with $u(q_a, q_b) = u(q_a, q_b) = f(u_a(q) + u_b(q_b))$ and $\psi(\pi_a, \pi_b) = g(\psi_a(\pi_a) + \psi_b(\pi_b))$. This would be a strong condition, with all terms involving ψ_{ab}^{*sj} and U_{ab}^{*js} vanishing (see Samuelson or Phlips). One might alternatively specify either $u(q_a, q_b)$ or $\psi(\pi_a, \pi_b)$ as strongly separable. The latter separability could also apply to the goods or subgroups of goods in groups a and/or b . For such cases, U_{aa} (and U_{aa}^{-1})

¹As in the case of the relative price version of the Rotterdam model (Theil, 1971), the mixed demand models developed here are not estimable without imposing restrictions on the basic model parameters ($\lambda\phi$, θ_i , θ_{ij} , γ_r , γ_m , ψ_{ab}^{*js} and u_{ab}^{*js}). Imposition of the general restrictions of adding up, homogeneity and symmetry is not sufficient for estimation; additional restrictions are needed.

and/or ψ (and ψ_{bb}^{-1}) are either diagonal or block diagonal, depending on the level of additivity, and θ_{ij} or γ_{rs} in (19) or (29) are restricted accordingly. For example, if U_{aa}^{-1} is block diagonal (strong separability between subgroups in group a), $\theta_{ij} = 0$ for i and j belonging to different subgroups.

Weak separability might also be useful as a source of restrictions. Suppose groups a and b are weakly separable in the direct utility function with $u(q_a, q_b) = u(u_a(q_a), u(q_b))$. The

cross-elasticity term U_{ab}^{*js} can then be written as $U_{ab}^{*js} = \frac{\partial^2 u}{\partial u_a \partial u_b} \frac{\partial u_a}{\partial q_{aj}} \frac{\partial u_b}{\partial q_{bs}} \frac{q_b}{\lambda \pi_{aj}}$

$$= \lambda \frac{\partial^2 u}{\partial u_a \partial u_b} \cdot \frac{w_{bs}}{\frac{\partial u}{\partial u_a} \frac{\partial u}{\partial u_b}}, \text{ where the first-order conditions have been used to obtain the last}$$

equality. The result suggests that if the budget shares are about equal across goods in group b , one might restrict U_{ab}^{*js} to be the same for all combinations of j in group a and s in group b . A similar result applies for weak separability of the indirect utility function, i.e., for $\psi(\pi_a, \pi_b)$

$= \psi(\psi_a(\pi_a), \psi_b(\pi_b))$, ψ_{ab}^{*sj} can be written as

$$\begin{aligned}\psi_{ab}^{*sj} &= \frac{\partial^2 \psi}{\partial \psi_a \partial \psi_b} \frac{\partial \psi_b}{\partial \pi_{bs}} \frac{\partial \psi_a}{\partial \pi_{aj}} \cdot \frac{\pi_{aj}}{\lambda q_{bs}} \\ &= \lambda \frac{\partial^2 \psi}{\partial \psi_a \partial \psi_b} \cdot \frac{w_{aj}}{\frac{\partial \psi}{\partial \psi_b} \frac{\partial \psi}{\partial \psi_a}}.\end{aligned}$$

One might not need to rely on separability to motivate restrictions involving groups a and b . For example, goods in group b might be viewed as having uniform substitute or complementary type impacts on the goods in group a , with the percentage change in the marginal utility of a good in group a , resulting from a one percent change in the quantity of a good in group b , being the same regardless of which good in group b is involved.

Weak separability can also be applied to obtain restrictions between goods or subgroups of goods in group a and/or b . In this case, U_{aa}^{ij} or ψ_{bb}^{rs} do not vanish for i and j belonging to different groups, or r and s belonging to different groups. However, the terms θ_{ij} or γ_{rs} can be restricted following the work by Theil for weak separability for the Rotterdam model. In short, for i and j (or r and s) belonging to different weakly separable groups, θ_{ij} (or γ_{rs}) equals a factor of proportionality times the product of θ_i and θ_j with the factor of proportionality dependent only on the two groups involved.

Mixed Demand Elasticities

We complete our analysis by noting the elasticities and flexibilities for model (19) or (29):

$$\varepsilon_{ai,aj} = \frac{\partial \log q_{ai}}{\partial \log \pi_{aj}} = \frac{\lambda \phi \left[\theta_{ij} - \theta_i \theta_j + \theta_i \sum_s \gamma_s \psi_{ab}^{*sj} \right]}{w_{ai}} - \frac{\theta_i}{w_{ai}} w_{aj}$$

$$\varepsilon_{ai,bs} = \frac{\partial \log q_{ai}}{\partial \log q_{bs}} = \frac{\lambda \phi \left[-\sum_j \theta_{ij} U_{ab}^{*js} + \theta_i \sum_j \theta_j U_{ab}^{*js} - \theta_i \gamma_s \right]}{w_{ai}} - \frac{\theta_i}{w_{ai}} w_{bs}$$

$$\varepsilon_{br,aj} = \frac{\partial \log \pi_{br}}{\partial \log \pi_{aj}} = \frac{\lambda \phi \left[-\sum_s \gamma_{rs} \psi_{ab}^{*sj} - \gamma_r \theta_j + \gamma_r \sum_s \gamma_s \psi_{ab}^{*sj} \right]}{w_{br}} - \frac{\gamma_r}{w_{br}} w_{aj}$$

$$\varepsilon_{br,bs} = \frac{\partial \log \pi_{br}}{\partial \log q_{bs}} = \frac{\lambda \phi \left[\gamma_{rs} + \gamma_r \sum_j \theta_j U_{ab}^{*js} - \gamma_r \gamma_s \right]}{w_{br}} - \frac{\gamma_r}{w_{br}} w_{bs}.$$

From (8), the income elasticities are

$$\varepsilon_{ai,x} = \frac{\partial \log q_{ai}}{\partial \log x} = -\sum_j \varepsilon_{ai,aj}$$

$$\varepsilon_{br,x} = \frac{\partial \log \pi_{br}}{\partial \log x} = -\sum_j \varepsilon_{br,aj},$$

while the scale elasticities are

$$\varepsilon_{ai,k} = \frac{\partial \log q_{ai}}{\partial \log x} = \sum_s \varepsilon_{ai,bs}$$

$$\varepsilon_{br,k} = \frac{\partial \log \pi_{br}}{\partial \log k} = \sum_s \varepsilon_{br,bs}.$$

The elasticities shown above are uncompensated. The first term on the right side of (19) involving $\sum_i w_{ai} d \log \pi_{ai} + \sum_s w_{bs} d \log q_{bs}$ is not the change in real income as discussed earlier.

Compensated elasticities can be obtained through the Hicksian demand equivalent of (19) as shown in Appendix B.

Concluding Comments

For demand studies where goods can be divided into quantity- and price-dependent groups, mixed demand system models may be useful. Moschini and Vissa, and Barten have proposed alternative mixed-demand modeling approaches which indicate how the basic restrictions of demand—adding-up, homogeneity and symmetry—can be consistently specified. Imposition of basic demand restrictions is generally useful for improving the precision of estimation, but additional restrictions are often needed to make the estimation problem more manageable, with separability typically utilized for this purpose.

In this study, the mixed demand problem has been examined using the fundamental equation approach, allowing mixed demand equations to be linked to the underlying direct and indirect utility functions. The linkage is important for specifying separability restrictions and other similar restrictions which depend on utility. As in Moschini and Vissa, and Barten, the analysis has focused on differential demand. A parameterization was chosen which directly relates mixed demand to the parameters of the quantity-dependent and price-dependent Rotterdam specifications. In addition, the differential mixed demand model can be straightforwardly extended to a levels model which may be useful for analysis of cross-sectional data or specifying dynamic relationships.

As in the case of the direct Rotterdam model where the link between model coefficients and the underlying utility function has been quite useful in specifying restrictions, the link between the mixed-demand model coefficients and the underlying direct and indirect utility functions may also prove to be useful for specifying restricted models for empirical analysis. The mixed-demand parameterization proposed allows one straightforwardly to develop and test various

separability and other restrictions on the Slutsky and Antonelli terms embedded in the specification.

Appendix A

Following Chavas, the conditional direct demand problem can be written as

$$(A1) \quad \begin{aligned} &\underset{q_a}{\text{maximize}} && u(q_a, q_b) \\ &\text{subject to} && \pi'_a q_a + \pi'_b q_b = 1 \end{aligned}$$

where π_a , π_b and q_b are predetermined.

The Lagrangean for problem (A1) is $\mathcal{L} = u(q_a, q_b) + \lambda_a (1 - \pi'_a q_a - \pi'_b q_b)$ and the

first-order conditions are

$$(A2) \quad \begin{aligned} \frac{\partial u}{\partial q_a} &= \lambda_a \pi_a \\ 1 &= \pi'_a q_a + \pi'_b q_b. \end{aligned}$$

The solution to (A2) yields conditional direct demand equations and conditional Lagrange multiplier

$$(A3) \quad \begin{aligned} q_a &= q_a(\pi_a, \pi_b, q_b) \\ \lambda_a &= \lambda_a(\pi_a, \pi_b, q_b). \end{aligned}$$

Total differentiation of first-order conditions (A2) results in

$$(A4) \quad \begin{bmatrix} U_{aa} & \pi_a \\ \pi'_a & 0 \end{bmatrix} \begin{bmatrix} dq_a \\ -d\lambda_a \end{bmatrix} = \begin{bmatrix} \lambda_a I & 0 & -U_{ab} \\ -q'_a & -q'_b & -\pi'_b \end{bmatrix} \begin{bmatrix} d\pi_a \\ d\pi_b \\ dq_b \end{bmatrix}.$$

Total differentiation of solution (A3) yields

$$(A5) \quad \begin{bmatrix} dq_a \\ -d\lambda_a \end{bmatrix} = \begin{bmatrix} \frac{\partial q_a}{\partial \pi'_a} & \frac{\partial q_a}{\partial \pi'_b} & \frac{\partial q_a}{\partial q'_b} \\ -\frac{\partial \lambda_a}{\partial \pi'_a} & -\frac{\partial \lambda_a}{\partial \pi'_b} & -\frac{\partial \lambda_a}{\partial q'_b} \end{bmatrix} \begin{bmatrix} d\pi_a \\ d\pi_b \\ dq_b \end{bmatrix}.$$

Combining (A4) and (A5) gives

$$(A6) \quad \begin{bmatrix} U_{aa} & \pi_a \\ \pi'_a & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial q_a}{\partial \pi'_a} & \frac{\partial q_a}{\partial \pi'_b} & \frac{\partial q_a}{\partial q'_b} \\ -\frac{\partial \lambda_a}{\partial \pi'_a} & -\frac{\partial \lambda_a}{\partial \pi'_b} & -\frac{\partial \lambda_a}{\partial q'_b} \end{bmatrix} = \begin{bmatrix} \lambda_a I & 0 & -U_{ab} \\ -q'_a & -q'_b & -\pi'_b \end{bmatrix},$$

or

$$\begin{aligned} \begin{bmatrix} \frac{\partial q_a}{\partial \pi'_a} & \frac{\partial q_a}{\partial \pi'_b} & \frac{\partial q_a}{\partial q'_b} \\ -\frac{\partial \lambda_a}{\partial \pi'_a} & -\frac{\partial \lambda_a}{\partial \pi'_b} & -\frac{\partial \lambda_a}{\partial q'_b} \end{bmatrix} &= \begin{bmatrix} U_{aa} & \pi_a \\ \pi'_a & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_a I & 0 & -U_{ab} \\ -q'_a & -q'_b & -\pi'_b \end{bmatrix} \\ &= (\pi'_a U_{aa}^{-1} \pi_a)^{-1} \begin{bmatrix} (\pi'_a U_{aa}^{-1} \pi_a) U_{aa}^{-1} - U_{aa}^{-1} \pi_a \pi'_a U_{aa}^{-1} & U_{aa}^{-1} \pi_a \\ \pi'_a U_{aa}^{-1} & -1 \end{bmatrix} \begin{bmatrix} \lambda_a I & 0 & -U_{ab} \\ -q'_a & -q'_b & -\pi'_b \end{bmatrix} \\ &= \begin{bmatrix} \lambda_a U_{aa}^{-1} - \lambda_a (\pi'_a U_{aa}^{-1} \pi_a)^{-1} U_{aa}^{-1} \pi_a \pi'_a U_{aa}^{-1} - (\pi'_a U_{aa}^{-1} \pi_a)^{-1} U_{aa}^{-1} \pi_a q'_a \\ \lambda_a (\pi'_a U_{aa}^{-1} \pi_a)^{-1} \pi'_a U_{aa}^{-1} + (\pi'_a U_{aa}^{-1} \pi_a)^{-1} q'_a \\ -(\pi'_a U_{aa}^{-1} \pi_a)^{-1} U_{aa}^{-1} \pi_a q'_b \\ (\pi'_a U_{aa}^{-1} \pi_a)^{-1} q'_b \\ -U_{aa}^{-1} U_{ab} + (\pi'_a U_{aa}^{-1} \pi_a)^{-1} U_{aa}^{-1} \pi_a \pi'_a U_{aa}^{-1} U_{ab} - (\pi'_a U_{aa}^{-1} \pi_a)^{-1} U_{aa}^{-1} \pi_a \pi'_b \\ -(\pi'_a U_{aa}^{-1} \pi_a)^{-1} \pi'_a U_{aa}^{-1} U_{ab} + (\pi'_a U_{aa}^{-1} \pi_a)^{-1} \pi'_b \end{bmatrix} \end{aligned}$$

Hence, the conditional income effects on demand are

$$\begin{aligned}
 (A7) \quad \frac{\partial q_a}{\partial x} &= -\frac{1}{x} \left[\frac{\partial q_a}{\partial \pi'_a} \pi_a + \frac{\partial q_a}{\partial \pi'_b} \pi_b \right] \\
 &= -\frac{1}{x} \left(\lambda_a U_{aa}^{-1} \pi_a - \lambda_a \left(\pi'_a U_{aa}^{-1} \pi_a \right)^{-1} U_{aa}^{-1} \pi_a \pi'_a U_{aa}^{-1} \pi_a - \left(\pi'_a U_{aa}^{-1} \pi_a \right)^{-1} U_{aa}^{-1} \pi_a q'_a \pi_a \right. \\
 &\quad \left. - \left(\pi'_a U_{aa}^{-1} \pi_a \right)^{-1} U_{aa}^{-1} \pi_a q'_b \pi_b \right) \\
 &= \frac{1}{x} \left(\pi'_a U_{aa}^{-1} \pi_a \right)^{-1} U_{aa}^{-1} \pi_a .
 \end{aligned}$$

The conditional inverse demand problem can be written as

$$\begin{aligned}
 (A8) \quad &\underset{\pi_b}{\text{maximize}} \quad -\psi(\pi_a, \pi_b) \\
 &\text{subject to} \quad \pi'_a q_a + \pi'_b q_b = 1 ,
 \end{aligned}$$

where q_a , q_b and π_a are predetermined.

The Lagrangean for problem (A8) is $\mathcal{L} = \psi(\pi_a, \pi_b) - \lambda_b (1 - \pi'_a q_a - \pi'_b q_b)$ and the first-order conditions are

$$\begin{aligned}
 (A9) \quad &-\frac{\partial \psi}{\partial \pi_b} = \lambda_b q_b \\
 &1 = \pi'_a q_a + \pi'_b q_b .
 \end{aligned}$$

The solution to (A9) is

$$\begin{aligned}
 (A10) \quad &\pi_b = \pi_b(q_a, q_b, \pi_a) \\
 &\lambda_b = \lambda_b(q_a, q_b, \pi_a) .
 \end{aligned}$$

Total differentiation of first-order conditions (A9) results in

$$(A11) \quad \begin{bmatrix} \psi_{bb} & q_b \\ q'_b & 0 \end{bmatrix} \begin{bmatrix} d\pi_b \\ -d\lambda_b \end{bmatrix} = \begin{bmatrix} \lambda_b I & 0 & -\psi_{ab} \\ -\pi'_b & -\pi'_a & -q'_a \end{bmatrix} \begin{bmatrix} dq_b \\ dq_a \\ d\pi_a \end{bmatrix}.$$

Total differentiation of solution (A10) yields

$$(A12) \quad \begin{bmatrix} d\pi_b \\ -d\lambda_b \end{bmatrix} = \begin{bmatrix} \frac{\partial \pi_b}{\partial q'_b} & \frac{\partial \pi_b}{\partial q'_a} & \frac{\partial \pi_b}{\partial \pi'_a} \\ -\frac{\partial \lambda_b}{\partial q'_b} & -\frac{\partial \lambda_b}{\partial q'_a} & -\frac{\partial \lambda_b}{\partial \pi'_a} \end{bmatrix} \begin{bmatrix} dq_b \\ dq_a \\ d\pi_a \end{bmatrix}.$$

Combining (A11) and (A12) gives

$$(A13) \quad \begin{bmatrix} \psi_{bb} & q_b \\ q'_b & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \pi_b}{\partial q'_b} & \frac{\partial \pi_b}{\partial q'_a} & \frac{\partial \pi_b}{\partial \pi'_a} \\ -\frac{\partial \lambda_b}{\partial q'_b} & -\frac{\partial \lambda_b}{\partial q'_a} & -\frac{\partial \lambda_b}{\partial \pi'_a} \end{bmatrix} = \begin{bmatrix} \lambda_b I & 0 & -\psi_{ab} \\ -\pi'_b & -\pi'_a & -q'_a \end{bmatrix}$$

or

$$\begin{aligned} & \begin{bmatrix} \frac{\partial \pi_b}{\partial q'_b} & \frac{\partial \pi_b}{\partial q'_a} & \frac{\partial \pi_b}{\partial \pi'_a} \\ -\frac{\partial \lambda_b}{\partial q'_b} & -\frac{\partial \lambda_b}{\partial q'_a} & -\frac{\partial \lambda_b}{\partial \pi'_a} \end{bmatrix} = \begin{bmatrix} \psi_{bb} & q_b \\ q'_b & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_b I & 0 & -\psi_{ab} \\ -\pi'_b & -\pi'_a & -q'_a \end{bmatrix} \\ & = \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} \begin{bmatrix} \left(q'_b \psi_{bb}^{-1} q_b \right) \psi_{bb}^{-1} - \psi_{bb}^{-1} q_b q'_b \psi_{bb}^{-1} & \psi_{bb}^{-1} q_b \\ q'_b \psi_{bb}^{-1} & -1 \end{bmatrix} \begin{bmatrix} \lambda_b I & 0 & -\psi_{ab} \\ -\pi'_b & -\pi'_a & -q'_a \end{bmatrix} \\ & = \begin{bmatrix} \lambda_b \psi_{bb}^{-1} - \lambda_b \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} \psi_{bb}^{-1} q_b q'_b \psi_{bb}^{-1} - \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} \psi_{bb}^{-1} q_b \pi'_b \\ \lambda_b \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} q'_b \psi_{bb}^{-1} + \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} \pi'_b \\ - \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} \psi_{bb}^{-1} q_b \pi'_a \\ \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} \pi'_a \end{bmatrix} \\ & \quad \begin{bmatrix} \psi_{bb}^{-1} \psi_{ab} - \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} \psi_{bb}^{-1} q_b q'_b \psi_{bb}^{-1} \psi_{ab} - \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} \psi_{bb}^{-1} q_b q'_a \\ \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} q'_b \psi_{bb}^{-1} \psi_{ab} + \left(q'_b \psi_{bb}^{-1} q_b \right)^{-1} q'_a \end{bmatrix}. \end{aligned}$$

The conditional scale effects on demand are

$$\begin{aligned} \text{(A14)} \quad k \frac{\partial \pi_b}{\partial k} &= \frac{\partial \pi_b}{\partial q_b'} q_b + \frac{\partial \pi_b}{\partial q_a'} q_a \\ &= -\left(q_b' \psi_{bb}^{-1} q_b\right)^{-1} \psi_{bb}^{-1} q_b \pi_b' q_b - \left(q_b' \psi_{bb}^{-1} q_b\right)^{-1} \psi_{bb}^{-1} q_b \pi_a' q_a \\ &= -\left(q_b' \psi_{bb}^{-1} q_b\right)^{-1} \psi_{bb}^{-1} q_b. \end{aligned}$$

Appendix B

Chavas discusses the general relationships between compensated and uncompensated effects in mixed demand systems. The compensation required in our mixed demand specification can be developed as follows. Consider the compensated mixed demand specification

$$(B1) \quad q_{ai} = q_{ai}^*(u, \pi_a, q_b),$$

where the first argument on the right side of the equation is utility u .

Differentiating (B1) with respect to the price of a good in group a results in the compensated or Slutsky type relationship

$$(B2) \quad \frac{\partial q_{ai}}{\partial \pi_{aj}} = \frac{\partial q_{ai}^*}{\partial \pi_{aj}} + \frac{\partial q_{ai}^*}{\partial u} \frac{\partial u}{\partial \pi_{aj}}$$

$$= \frac{\partial q_{ai}^*}{\partial \pi_{aj}} + \frac{\frac{\partial q_{ai}}{\partial x}}{\frac{\partial u}{\partial x}} \frac{\partial u}{\partial \pi_{aj}}$$

or, in terms of elasticities,

$$\frac{\partial q_{ai}}{\partial \pi_{aj}} \frac{\pi_{aj}}{q_{ai}} = \frac{\partial q_{ai}^*}{\partial \pi_{aj}} \frac{\pi_{aj}}{q_{ai}} + \frac{\partial q_{ai}}{\partial x} \frac{x}{q_{ai}} \frac{\frac{\partial u}{\partial \pi_{aj}} \frac{\pi_{aj}}{u}}{\frac{\partial u}{\partial x} \frac{x}{u}}$$

or

$$\epsilon_{ai,aj} = \epsilon_{ai,aj}^* + \epsilon_{ai,x} \frac{\frac{\partial \log u}{\partial \log \pi_{aj}}}{\frac{\partial \log u}{\partial \log x}}.$$

Noting that $u = u(q_a(\pi_a, q_b), q_b)$ and recalling the first-order condition $\frac{\partial u}{\partial q_{ai}} = \lambda \pi_{ai}$,

the term $\frac{\partial u}{\partial \pi_{aj}}$ can be written as

$$\begin{aligned} \text{(B3)} \quad \frac{\partial u}{\partial \pi_{aj}} &= \sum_i \frac{\partial u}{\partial q_{ai}} \frac{\partial q_{ai}}{\partial \pi_{aj}} \\ &= \lambda \sum_i \pi_{ai} \frac{\partial q_{ai}}{\partial \pi_{aj}} \end{aligned}$$

or, in terms of elasticities,

$$\frac{\partial u}{\partial \pi_{aj}} \frac{\pi_{aj}}{u} = \frac{\lambda}{u} \sum_i \pi_{ai} q_{ai} \frac{\partial q_{ai}}{\partial \pi_{aj}} \frac{\pi_{aj}}{q_{ai}}$$

or

$$\frac{\partial \log u}{\partial \log \pi_{aj}} = \frac{\lambda}{u} \sum_i w_{ai} \varepsilon_{ai,aj}.$$

Likewise, the term $\frac{\partial u(q_a(\pi_a, q_b), q_b)}{\partial x}$ can be written as

$$\begin{aligned} \text{(B4)} \quad \frac{\partial u}{\partial x} &= \sum_i \frac{\partial u}{\partial q_{ai}} \frac{\partial q_{ai}}{\partial x} \\ &= \lambda \sum_i \pi_{ai} \frac{\partial q_{ai}}{\partial x} \end{aligned}$$

or, in terms of elasticities,

$$\frac{\partial u}{\partial x} \frac{x}{u} = \frac{\lambda}{u} \sum_i \pi_{ai} q_{ai} \frac{\partial q_{ai}}{\partial x} \frac{x}{q_{ai}}$$

or

$$\frac{\partial \log u}{\partial \log x} = \frac{\lambda}{u} \sum_i w_{ai} \varepsilon_{ai,x}.$$

Substituting results (B3) and (B4) into (B2), we find the Slutsky relationships can be written as

$$(B5) \quad \varepsilon_{ai,aj} = \varepsilon_{ai,aj}^* + \varepsilon_{ai,x} \frac{\sum_i w_{ai} \varepsilon_{ai,aj}}{\sum_i w_{ai} \varepsilon_{ai,x}}.$$

From (10a), note that differentiation of the budget constraint $\pi'_a q_a(\pi_a, q_b) + \pi_b(\pi_a, q_b)' q_b = 1$ with respect to π_{aj} can be written as

$$(B6) \quad q_{aj} + \sum_i \pi_{ai} \frac{\partial q_{ai}}{\partial \pi_{aj}} + \sum_r q_{br} \frac{\partial \pi_{br}}{\partial \pi_{aj}} = 0$$

or, after multiplying through by π_{aj}

$$\pi_{aj} q_{aj} + \sum_i \pi_{ai} q_{ai} \frac{\partial q_{ai}}{\partial \pi_{aj}} \frac{\pi_{aj}}{q_{ai}} + \sum_r q_{br} \pi_{br} \frac{\partial \pi_{br}}{\partial \pi_{aj}} \frac{\pi_{aj}}{\pi_{br}} = 0$$

or

$$\sum_i w_{ai} \varepsilon_{ai,aj} = -w_{aj} - \sum_r w_{br} \varepsilon_{br,aj}.$$

Likewise, from (8) and (10c) differentiation of the budget constraint with respect to income can be written as

$$(B7) \quad -\frac{1}{x} \sum_i \pi_{ai} q_{ai} + \sum_i \pi_{ai} \frac{\partial q_{ai}}{\partial x} + \sum_r q_{br} \frac{\partial \pi_{br}}{\partial x} = 0$$

or, in terms of elasticities

$$\frac{1}{x} \left[-\sum_i w_{ai} + \sum_i \pi_{ai} q_{ai} \frac{\partial q_{ai}}{\partial x} \frac{x}{q_{ai}} + \sum_r q_{br} \pi_{br} \frac{\partial \pi_{br}}{\partial x} \frac{x}{\pi_{br}} \right] = 0$$

or

$$\sum w_{ai} \varepsilon_{ai,x} = \sum_i w_{ai} - \sum_r w_{br} \varepsilon_{br,x}.$$

Substituting results (B6) and (B7) into (B5), the Slutsky relationship can also be written as

$$(B8) \quad \varepsilon_{ai,aj} = \varepsilon_{ai,aj}^* - \varepsilon_{ai,x} \frac{\left(w_{aj} + \sum_r w_{br} \varepsilon_{br,aj} \right)}{\left(\sum_i w_{ai} - \sum_r w_{br} \varepsilon_{br,x} \right)}.$$

Result (B8) allows a straightforward comparison between the Slutsky relationship for the mixed demand problem and the Slutsky relationship for the usual direct specification of demand.

If all goods are in group a , (B8) becomes

$$(B9) \quad \varepsilon_{ij} = \varepsilon_{ij}^* - \varepsilon_{i,x} w_j,$$

where the group subscript has been dropped. Result (B9), of course, is the usual Slutsky

relationship for the direct specification of demand. From (B2), we see that $-w_j = \frac{\left[\frac{\partial \log u}{\partial \log \pi_j} \right]}{\left[\frac{\partial \log u}{\partial \log x} \right]},$

which we know as Roy's identity. When there are goods in group b , the compensation term

$\frac{\left[\frac{\partial \log u}{\partial \log \pi_{aj}} \right]}{\left[\frac{\partial \log u}{\partial \log x} \right]}$ is no longer equal to w_{aj} in general. Instead, a change in π_{aj} involves effects

through the prices in group b , the π_{br} 's.

The Slutsky relationship for the effect of q_{bs} on q_{ai} can be similarly developed, i.e.,

$$(B10) \quad \frac{\partial q_{ai}}{\partial q_{bs}} = \frac{\partial q_{ai}^*}{\partial q_{bs}} + \frac{\partial q_{ai}^*}{\partial u} \frac{\partial u}{\partial q_{bs}}$$

$$= \frac{\partial q_{ai}^*}{\partial q_{bs}} + \frac{\frac{\partial q_{ai}}{\partial x}}{\frac{\partial u}{\partial x}} \frac{\partial u}{\partial q_{bs}}$$

or, in terms of elasticities

$$\frac{\partial q_{ai}}{\partial q_{bs}} \frac{q_{bs}}{q_{ai}} = \frac{\partial q_{ai}^*}{\partial q_{bs}} \frac{q_{bs}}{q_{ai}} + \frac{\partial q_{ai}}{\partial x} \frac{x}{q_{ai}} \frac{\frac{\partial u}{\partial q_{bs}} \frac{q_{bs}}{u}}{\frac{\partial u}{\partial x} \frac{x}{u}}$$

or

$$\epsilon_{ai,bs} = \epsilon_{ai,bs}^* + \epsilon_{ai,x} \frac{\frac{\partial \log u}{\partial \log q_{bs}}}{\frac{\partial \log u}{\partial \log x}}.$$

Given $u = u(q_a(\pi_a, q_b), q_b) = \psi(\pi_a, \pi_b(\pi_a, q_b))$ and first-order condition $\frac{\partial \psi}{\partial \pi_{br}}$

$= \lambda q_{br}$, the term $\frac{\partial u}{\partial q_s}$ can be written as

$$(B11) \quad \frac{\partial u}{\partial q_{bs}} = \sum_r \frac{\partial \psi}{\partial \pi_{br}} \frac{\partial \pi_{br}}{\partial q_{bs}}$$

$$= \lambda \sum_r q_{br} \frac{\partial \pi_{br}}{\partial q_{bs}}$$

or, in terms of elasticities

$$\frac{\partial u}{\partial q_{bs}} \frac{q_{bs}}{u} = \frac{\lambda}{u} \sum_r q_{br} \pi_{br} \frac{\partial \pi_{br}}{\partial q_{bs}} \cdot \frac{q_{bs}}{\pi_{br}}$$

or

$$\frac{\partial \log u}{\partial \log q_{bs}} = \frac{\lambda}{u} \sum_r w_{br} \varepsilon_{br,bs}.$$

Substituting (B4) and (B11) into (B10), Slutsky relationship (B10) can be written as

$$(B12) \quad \varepsilon_{ai,bs} = \varepsilon_{ai,bs}^* + \varepsilon_{ai,x} \frac{\sum_r w_{br} \varepsilon_{br,bs}}{\sum_i w_{ai} \varepsilon_{ai,x}}.$$

From (10b), we see that differentiation of the budget constraint with respect to q_{bs} results in

$$(B13) \quad \sum_i \pi_{ai} \frac{\partial q_{ai}}{\partial q_{bs}} + \sum_r q_{br} \frac{\partial \pi_{br}}{\partial q_{bs}} + \pi_{bs} = 0$$

or, after multiplying through by q_{bs}

$$\sum_i \pi_{ai} q_{ai} \frac{\partial q_{ai}}{\partial q_{bs}} \frac{q_{bs}}{q_{ai}} + \sum_r q_{br} \pi_{br} \frac{\partial \pi_{br}}{\partial q_{bs}} \frac{q_{bs}}{\pi_{br}} + \pi_{bs} q_{bs} = 0$$

or

$$\sum_r w_{br} \varepsilon_{br,bs} = -w_{bs} - \sum_i w_{ai} \varepsilon_{ai,bs}.$$

Substituting (B7) and (B13) into (B12), we find

$$(B14) \quad \varepsilon_{ai,bs} = \varepsilon_{ai,bs}^* - \varepsilon_{ai,x} \frac{\left(w_{bs} + \sum_i w_{ai} \varepsilon_{ai,bs} \right)}{\left(\sum_i w_{ai} - \sum w_{br} \varepsilon_{br,x} \right)}.$$

For the price-dependent equations of the mixed demand system, we can develop Slutsky relationships from compensated specification

$$(B15) \quad \pi_{br} = \pi_{br}^*(u, \pi_a, q_b).$$

Differentiation of (B15) with respect to π_{aj} results in

$$(B16) \quad \begin{aligned} \frac{\partial \pi_{br}}{\partial \pi_{aj}} &= \frac{\partial \pi_{br}^*}{\partial \pi_{aj}} + \frac{\partial \pi_{br}^*}{\partial u} \frac{\partial u}{\partial \pi_{aj}} \\ &= \frac{\partial \pi_{br}^*}{\partial \pi_{aj}} + \frac{\frac{\partial \pi_{br}^*}{\partial x}}{\frac{\partial u}{\partial x}} \frac{\partial u}{\partial \pi_{aj}} \end{aligned}$$

or, in terms of elasticities

$$\frac{\partial \pi_{br}}{\partial \pi_{aj}} \frac{\pi_{aj}}{\pi_{br}} = \frac{\partial \pi_{br}^*}{\partial \pi_{aj}} \frac{\pi_{aj}}{\pi_{br}} + \frac{\partial \pi_{br}^*}{\partial x} \frac{x}{\pi_{br}} \frac{\frac{\partial u}{\partial \pi_{aj}} \frac{\pi_{aj}}{u}}{\frac{\partial u}{\partial x} \frac{x}{u}}$$

or

$$\varepsilon_{br,aj} = \varepsilon_{br,aj}^* + \varepsilon_{br,x} \frac{\frac{\partial \log u}{\partial \log \pi_{aj}}}{\frac{\partial \log u}{\partial \log x}}$$

or, as we found in obtaining result (B8)

$$\varepsilon_{br,aj} = \varepsilon_{br,aj}^* + \varepsilon_{br,x} \frac{\left(w_{aj} + \sum_r w_{br} \varepsilon_{br,aj} \right)}{\left(\sum_i w_{ai} - \sum_r w_{br} \varepsilon_{br,x} \right)}.$$

Likewise, differentiation of (B15) with respect to q_{bs} results in

$$\begin{aligned} \text{(B17)} \quad \frac{\partial \pi_{br}}{\partial q_{bs}} &= \frac{\partial \pi_{br}^*}{\partial q_{bs}} + \frac{\partial \pi_{br}^*}{\partial u} \frac{\partial u}{\partial q_{bs}} \\ &= \frac{\partial \pi_{br}^*}{\partial q_{bs}} + \frac{\frac{\partial \pi_{br}}{\partial x}}{\frac{\partial u}{\partial x}} \frac{\partial u}{\partial q_{bs}} \end{aligned}$$

or, in terms of elasticities

$$\frac{\partial \pi_{br}}{\partial q_{bs}} \frac{q_{bs}}{\pi_{br}} = \frac{\partial \pi_{br}^*}{\partial q_{bs}} \frac{q_{bs}}{\pi_{br}} + \frac{\partial \pi_{br}}{\partial x} \frac{x}{\pi_{br}} \frac{\frac{\partial u}{\partial q_{bs}} \frac{q_{bs}}{u}}{\frac{\partial u}{\partial x} \frac{x}{u}}$$

or

$$\varepsilon_{br,bs} = \varepsilon_{br,bs}^* + \varepsilon_{br,x} \frac{\frac{\partial \log u}{\partial \log q_{bs}}}{\frac{\partial \log u}{\partial \log x}}$$

or, as found in obtaining result (B14)

$$\varepsilon_{br,bs} = \varepsilon_{br,bs}^* + \varepsilon_{br,x} \frac{\left(w_{bs} + \sum_i w_{ai} \varepsilon_{ai,bs} \right)}{\left(\sum_i w_{ai} - \sum_r w_{br} \varepsilon_{br,x} \right)}.$$

Compensated results (B16) and (B17) also can be written in alternative forms which can be directly compared to the usual compensated results for inverse demand systems. For example, based on the scale effect, result (B17) can be written as

$$(B18) \quad \frac{\partial \pi_{br}}{\partial q_{bs}} = \frac{\partial \pi_{br}^*}{\partial q_{bs}} + \frac{\partial \pi_{br}^*}{\partial u} \frac{\partial u}{\partial q_{bs}}$$

$$= \frac{\partial \pi_{br}^*}{\partial q_{bs}} + \frac{\frac{\partial \pi_{br}}{\partial k}}{\frac{\partial u}{\partial k}} \frac{\partial u}{\partial q_{bs}}$$

or, in terms of elasticities,

$$\frac{\partial \pi_{br}}{\partial q_{bs}} \cdot \frac{q_{bs}}{\pi_{br}} = \frac{\partial \pi_{br}^*}{\partial q_{bs}} \frac{q_{bs}}{\pi_{br}} + \frac{\partial \pi_{br}}{\partial k} \frac{k}{\pi_{br}} \frac{\frac{\partial u}{\partial q_{bs}} \frac{q_{bs}}{u}}{\frac{\partial u}{\partial k} \frac{k}{u}}$$

or

$$\varepsilon_{br,bs} = \varepsilon_{br,bs}^* + \varepsilon_{br,k} \frac{\frac{\partial \log u}{\partial \log q_{bs}}}{\frac{\partial \log u}{\partial \log k}}.$$

Given $u = \psi(\pi_a, \pi_b(\pi_a, q_b))$ and first-order condition $\frac{\partial \psi}{\partial \pi_{br}} = \lambda q_{br}$, the term $\frac{\partial u}{\partial k}$ can

be written as

$$(B19) \quad \frac{\partial u}{\partial k} = \sum_r \frac{\partial \psi}{\partial \pi_{br}} \frac{\partial \pi_{br}}{\partial k}$$

$$= \lambda \sum_r q_{br} \frac{\partial \pi_{br}}{\partial k}$$

or, in terms of elasticities

$$\frac{\partial u}{\partial k} \frac{k}{u} = \frac{\lambda}{u} \sum_r q_{br} \pi_{br} \frac{\partial \pi_{br}}{\partial k} \frac{k}{\pi_{br}}$$

or

$$\frac{\partial \log u}{\partial \log k} = \frac{\lambda}{u} \sum_r w_{br} \varepsilon_{br,k}.$$

Substituting results (B11) and (B19) into (B18) results in

$$(B20) \quad \varepsilon_{br,bs} = \varepsilon_{br,bs}^* + \varepsilon_{br,k} \frac{\sum_r w_{br} \varepsilon_{br,bs}}{\sum_r w_{br} \varepsilon_{br,k}}.$$

From (10d), differentiation of the budget constraint with respect to the scale parameter k can be written as

$$(B21) \quad \sum_i \pi_{ai} \frac{\partial q_{ai}}{\partial k} + \sum_r q_{br} \frac{\partial \pi_{br}}{\partial k} + \sum_r \pi_{br} q_{br}^0 = 0$$

or, in terms of elasticities

$$\sum_i \pi_{ai} q_{ai} \frac{\partial q_{ai}}{\partial k} \frac{k}{q_{ai}} + \sum_r q_{br} \pi_{br} \frac{\partial \pi_{br}}{\partial k} \frac{k}{\pi_{br}} + \sum_r \pi_{br} q_{br} = 0$$

or

$$\sum_r w_{br} \varepsilon_{br,k} = -\sum_r w_{br} - \sum_i w_{ai} \varepsilon_{ai,k}$$

Substituting (B13) and (B21) into (B20) yields

$$(B22) \quad \varepsilon_{br,bs} = \varepsilon_{br,bs}^* + \varepsilon_{br,k} \frac{w_{bs} + \sum_i w_{ai} \varepsilon_{ai,bs}}{\sum_r w_{br} + \sum_i w_{ai} \varepsilon_{ai,k}}.$$

Like (B8), result (B22) allows a straightforward comparison between the compensated relationship for the mixed demand specification and the compensated relationship for the usual inverse specification of demand. If all goods are in group b , (B22) becomes

$$(B23) \quad \varepsilon_{r,s} = \varepsilon_{r,s}^* + \varepsilon_{r,k} w_s.$$

where the group subscript has been dropped. Result (B23) is the usual compensated relationship

$$\text{for the inverse specification of demand. From (B18), we see that } w_s = \frac{\left[\frac{\partial \log u}{\partial \log q_s} \right]}{\left[\frac{\partial \log u}{\partial \log k} \right]}, \text{ which we}$$

know as Wold's identity. When there are goods in both groups a and b , the compensation

$$\text{term } \frac{\left[\frac{\partial \log u}{\partial \log q_{bs}} \right]}{\left[\frac{\partial \log u}{\partial \log k} \right]} \text{ is no longer equal to } w_{bs} \text{ but involves effects through the quantities in group}$$

a , the q_{ai} 's.

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